International Journal of Physical Sciences and Research

GLINT OPEN ACCESS

Volume 1, Issue 1
Research Article

Date of Submission: 29 October, 2025
Date of Acceptance: 02 December, 2025
Date of Publication: 15 December, 2025

Semi-Analytical Analysis of Thin Plate Deflections Using the Clebsch–Ritz Method

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Citation: Hemza, G., Salah, G., Bouchelaghem, H., Hanene, B., Athmane, Y, M. (2025). Semi-Analytical Analysis of Thin Plate Deflections Using the Clebsch–Ritz Method. *Int J Phys Sci Res*. 1(1), 01-12.

Abstract

Plates of various geometries and materials are widely employed in structural and mechanical engineering, where accurate prediction of their deformation behavior is crucial for design and safety. This study presents a semi-analytical approach for determining the deflection of thin rectangular plates resting on an elastic foundation with free boundary conditions. The proposed method integrates Clebsch's analytical solution of the plate deformation equation with the Ritz energy method to compute the unknown coefficients governing the displacement field. By combining the strain energy of deformation with the external work of the applied load, the governing variational equations are established and solved to determine the deformation parameters. The developed formulation allows direct evaluation of the plate deflection at any point for arbitrary loading conditions. Numerical implementation using *Mathematica* validates the accuracy of the model and confirms its efficiency in representing the physical behavior of plates on elastic media. The proposed Clebsch–Ritz approach offers a reliable and adaptable framework for analyzing static deformation problems in structural and mechanical applications.

Keywords: Plate's Deflections, Ritz's Method, Clebsch's Solution

Introduction

The rectangular plates are applied in different fields such as construction of civil, naval, aeronautics, and space structures. A better understanding of the vibratory behavior of plates is desirable to have a set of expectation of static and dynamic behavior for design. and also the advanced characteristics of these structures in order to better commissioning. Therefore, analysis of plates of different types is a hot topic.

Historically, Euler was one of the first in the 18th century who gave a second-order differential equation to define the vibration of plates assimilated to an elastic membrane formed by orthogonally crossed elastic wires [1]. The German physicist Chladni, in 1787, discovered the first proper modes of vibration of a square plate by covering the vibrating surface with powder [2]. Lagrange developed in 1811 the first correct differential equation to describe the vibrations of a free plate of constant thickness, in which the flexion w must be satisfied [1]. Sophie Germain was rewarded in 1816 for her development of the thin plate equation by adding a term to the Euler equation taking into account the radii of curvature in both perpendicular directions [1]. Sometime later, Navier used the trigonometric functions discovered by Fourier to represent the deformation of a plate for certain boundary conditions [1]. The Kirchhoff–Love theory of plates is a two-dimensional mathematical model that is used to determine the stresses and deformations in thin plates subjected to forces and moments [3]. This theory is an extension of Euler-Bernoulli beam theory and was developed in 1888 by Love using assumptions proposed by Kirchhoff. The theory assumes that a mid-surface plane can be used to represent a three-dimensional plate in two-dimensional form. Lord-Rayleigh or Ritz were improved the plate's analysis procedure based on the shape functions that characterize the eigenvalue movement, where each function with an independent amplitude coefficient [4,5]. This procedure is called: Rayleigh Ritz Method or simply Ritz Method. It's the most approximate method used in the analysis of structures' vibrations. In 1921, Timoshenko shows that taking into account the respective effects of rotational inertia and shear have the effect of significantly reducing the natural vibration frequencies of the beams [1-6]. In this context, vibratory study of the plates evolved by injecting additional hypotheses such as the shearing effect for thick plates. These efforts led to more refined models like that of Love who investigated the application Kirchhoff's work to thick plates [7,8]. Since then, numerous studies have been conducted, especially with the development of civil, naval and aeronautical structures. Reissner was one of the first to extend shear plate theory in the static case [9]. Then, Uflyand [10], Mindlin [8], Waburton[11], Leissa [12] and others followed this expanded computational principle by considering additional factors to achieve more realistic results [8,10-12]. Such contributions allows to better understanding the plate vibration problem and taking into account the various hypotheses in order to be able to reach the exact dynamic behavior of the plates as much as possible [13]. Leissa considered the free vibrations of a rectangular plates with different boundary conditions in his works [14-15]. Finally, the authors of the work give the abilities of the Ritz's method in the other applications like the study of the contact problems [16]. So, our goal is to get a solution in semi analytical form to get more precision of the results.

Problem

This paper considers the determination of the deflections of a thin rectangular plate with a free boundary conditions over its entire contour and resting on the surface of an elastic foundation.

The differential equation of the plate's deflections is given by [1]:

$$\Delta\Delta W(x,y) = \frac{q(x,y)}{D} \tag{1}$$

Where,

$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$: Laplace operator;

$W(x,y)$: function of displacements of the plate;

$q(x,y)$: function of the external load subjected on the plate;

D: Cylindrical rigidity of the plate.

To solve the differential equation of order 4, given by the equation (1), Clebsch's solution [1] is adopted given in the form of equation (2) for the general case [17]:

$$W(x,y) = W_0(x,y) + \sum_{n=1}^{\infty} A_n W_n(x,y) \tag{2}$$

For the case considered herein, the continuous contact between the plate and the surface of elastic foundation is replaced by a plate with an entirely free contour and a fictional embedment at a point coinciding with the plate's center. Therefore, the terms of equation (2) satisfying the boundary conditions of the studied plate are as per equation (3) [17]:

$$W(x,y) = W_0(x,y) + A_{22}W_1(x,y) + B_{22}W_2(x,y) + A_{31}W_3(x,y) + B_{31}W_4(x,y) + \ldots \tag{3}$$

Where,

$W_0(x,y)$: a special solution for this case given by [17]:

$$W_0(x,y) = \frac{\mathrm{Pab}}{16\pi D}\left\{\left[\left(\frac{x}{a}-\frac{t}{a}\right)^2 + \left(\frac{y}{b}-\frac{z}{b}\right)^2\right]\ln\left[\left(\frac{x}{a}-\frac{t}{a}\right)^2 + \left(\frac{y}{b}-\frac{z}{b}\right)^2\right] + 4\left(\frac{\mathrm{xt}}{a^2}+\frac{\mathrm{yz}}{b^2}\right)\left(1+\ln\left[\frac{\sqrt{x^2+y^2}\sqrt{t^2+z^2}}{\mathrm{ab}}\right]\right) - \left(\frac{t^2}{a^2}+\frac{z^2}{b^2}\right)\ln\left[\frac{t^2}{a^2}+\frac{z^2}{b^2}\right] - \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\ln\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}\right]\right\}$$

P: External load applied at any point of the plate;

a and b: length and width of the plate, respectively;

t and z: coordinates of the point where the external load is applied;

According to [17]:

$$W_1(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}\ ;$$
$$W_2(x,y)=\frac{2\mathrm{xy}}{\mathrm{ab}}\ ;$$
$$W_3(x,y)=\frac{x}{a}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\ ;$$
$$W_4(x,y)=\frac{\mathrm{y}}{\mathrm{b}}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\ ;$$

$D=\frac{Eh^3}{12(1-v^2)}$: cylindrical rigidity of the plate;
E, v: elasticity modulus and Poisson's ratio of the plate's materials, respectively;
A_{22}, B_{22}, A_{31} and B_{31} coefficients determined by the application of the Ritz energy method.

Application of The Energy Method of Ritz

The coefficients A_{22}, B_{22}, A_{31} and B_{31} relate to the Clebsch solution and are determined by the application of the principle variation of energy of the Ritz's method. This comes from the combination of the variation of the energy of the deformation of the plate and the work of the deformation of the plate during the passage of the plate from the deformed state into the initial state [1-17].

The deformation energy U of the plate is given by [17] as per equation (5):

$$U=\frac{D}{2}\int\int_{\Omega}\psi(\mathrm{W}(x,y))\,\mathrm{d}x\,\mathrm{d}y \tag{5}$$

Where,

$$\psi(x,y)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-\beta\left[\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}-\left(\frac{\partial^2}{\partial x\,\partial y}\right)^2\right];$$
$$\beta=2(1-\mathrm{v})\ ;$$

The work of deformation of the plate during the passage of the plate from the deformed state into the initial state is given by [17] as per equation (6)

$$\Pi=A_{22}W_1(t,z)+B_{22}W_2(t,z)+A_{31}W_3(t,z)+B_{31}W_4(t,z)+\ldots \tag{6}$$

where,

$W_1(t,z), W_2(t,z), W_3(t,z)$ and $W_4(t,z)$ defined by the same formulas above where the variables (x,y) are replaced by the variables (t,z).

The total deformation energy of the plate is the combination of the formulas (5) and (6), i.e.:

$$\ni = U(x,y) + \Pi(t,z) \tag{7}$$

Finally, the coefficients A_{22}, B_{22}, A_{31} and B_{31} are determined using the principle of the least square's method [18], i.e.:

$$\begin{cases} \dfrac{\partial \ni}{\partial A_{22}} = 0; \\ \dfrac{\partial \ni}{\partial B_{22}} = 0; \\ \dfrac{\partial \ni}{\partial A_{31}} = 0; \\ \dfrac{\partial \ni}{\partial B_{31}} = 0; \end{cases} \tag{8}$$

Resolving the matrix system (8) permits determination of the formula's coefficients' in a final form. Note that the terms of the matrix system (9) are in a long and very complicated form, makes their calculation and/or simplification quite difficult.

$$\begin{pmatrix} \frac{16D\beta}{a^2} & 0 & 0 & 0 \\ 0 & \frac{16D\beta}{a^2} & 0 & 0 \\ 0 & 0 & \frac{-32D(\beta-8)}{3a^2} & 0 \\ 0 & 0 & 0 & \frac{-32D(\beta-8)}{3a^2} \end{pmatrix} \begin{pmatrix} A_{22} \\ B_{22} \\ A_{31} \\ B_{31} \end{pmatrix} = \begin{pmatrix} -Q_{11} + \left(\frac{t^2-z^2}{a^2}\right) \\ -Q_{22} + \left(\frac{2tz}{a^2}\right) \\ -Q_{33} + t\left(\frac{t^2+z^2}{a^3}\right) \\ -Q_{44} + z\left(\frac{t^2+z^2}{a^3}\right) \end{pmatrix} \tag{9}$$

$$Q_1 = \frac{1}{8a^2\pi}\beta\Big(2xz + 2x(-2t+x)\arctan\left[\frac{x}{y}\right] - 2y(y-2z)\arctan\left[\frac{y}{x}\right] + 2x(-2t+x)\arctan\left[\frac{t-x}{y-z}\right] -$$
$$-\left(t^2+(y-z)^2\right)\arctan\left[\frac{y-z}{t-x}\right] + \left(t^2+(y-z)^2\right)\arctan\left[\frac{-y+z}{t-x}\right]\Big);$$

$$Q_2 = \frac{1}{8a^2\pi}\beta\Big(y^2-(y-z)^2 + 2x(-\mathrm{i}t+z)\arctan\left[\frac{x}{y}\right] + 2x(\mathrm{i}t+z)\arctan\left[\frac{x}{y}\right] + 4xz\arctan\left[\frac{y}{x}\right] -$$
$$-x^2\log\left[x^2+y^2\right] - y(y-2z)\log\left[x^2+y^2\right] + x(t-\mathrm{i}z)\log\left[x^2+y^2\right] + x(t+\mathrm{i}z)\log\left[x^2+y^2\right] +$$
$$+(t-x)^2\log\left[t^2-2tx+x^2+(y-z)^2\right] + (y-z)^2\log\left[t^2-2tx+x^2+(y-z)^2\right]\Big);$$

$$
\begin{aligned}
Q_3 = -\frac{1}{24a^3\pi}\Big(& -16txy - 32txz + 16x^2z + 4txy\beta + 8txz\beta - 4x^2z\beta - 2\left(-4x^3(-4+\beta) + 3t\left(x^2+y^2\right)(-8+3\beta)\right)\arctan\left[\frac{x}{y}\right] + \\
& + 2x\left(3tx(8-3\beta) + 4x^2(-4+\beta) + 6t^2\beta\right)\arctan\left[\frac{t-x}{y-z}\right] + 16t^3\arctan\left[\frac{y-z}{t-x}\right] - 48ty^2\arctan\left[\frac{y-z}{t-x}\right] + 96tyz\arctan\left[\frac{y-z}{t-x}\right] - \\
& - 48tz^2\arctan\left[\frac{y-z}{t-x}\right] + 2t^3\beta\arctan\left[\frac{y-z}{t-x}\right] + 18ty^2\beta\arctan\left[\frac{y-z}{t-x}\right] - 36tyz\beta\arctan\left[\frac{y-z}{t-x}\right] + 18tz^2\beta\arctan\left[\frac{y-z}{t-x}\right] - \\
& - 16y^3\log\left[1+\frac{x^2}{y^2}\right] + 4y^3\beta\log\left[1+\frac{x^2}{y^2}\right] + 24y^3\log\left[x^2+y^2\right] - 24x^2z\log\left[x^2+y^2\right] - 24y^2z\log\left[x^2+y^2\right] - \\
& - 6y^3\beta\log\left[x^2+y^2\right] + 9x^2z\beta\log\left[x^2+y^2\right] + 9y^2z\beta\log\left[x^2+y^2\right] + 24x^2y\log\left[\frac{x^2+y^2}{a^2}\right] - 6x^2y\beta\log\left[\frac{x^2+y^2}{a^2}\right] + \\
& + 24t^2y\log\left[t^2-2tx+x^2+(y-z)^2\right] - 8y^3\log\left[t^2-2tx+x^2+(y-z)^2\right] - 24t^2z\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 24x^2z\log\left[t^2-2tx+x^2+(y-z)^2\right] + 24y^2z\log\left[t^2-2tx+x^2+(y-z)^2\right] - 24yz^2\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 8z^3\log\left[t^2-2tx+x^2+(y-z)^2\right] - 6t^2y\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + 2y^3\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 3t^2z\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + 6txz\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - 9x^2z\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - \\
& - 9y^2z\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + 12yz^2\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - 5z^3\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - \\
& - 24x^2y\log\left[\frac{t^2-2tx+x^2+(y-z)^2}{a^2}\right] + 6x^2y\beta\log\left[\frac{t^2-2tx+x^2+(y-z)^2}{a^2}\right]\Big);
\end{aligned}
$$

$$
\begin{aligned}
Q_4 = \frac{1}{24a^3\pi}\Big(& -16t^2x - 8tx^2 + 16xyz + 16xz^2 + 10t^2x\beta + 2tx^2\beta - 4xyz\beta + 2xz^2\beta + 2\left(3x^2z(8-3\beta) + 3y^2z(8-3\beta) + 4y^3(-4+\beta)\right)\times \\
& \times\arctan\left[\frac{x}{y}\right] + 6(2t-x)xz(-8+3\beta)\arctan\left[\frac{t-x}{y-z}\right] + 32y^3\arctan\left[\frac{y-z}{t-x}\right] - 48t^2z\arctan\left[\frac{y-z}{t-x}\right] - 48y^2z\arctan\left[\frac{y-z}{t-x}\right] + \\
& + 16z^3\arctan\left[\frac{y-z}{t-x}\right] - 8y^3\beta\arctan\left[\frac{y-z}{t-x}\right] + 18t^2z\beta\arctan\left[\frac{y-z}{t-x}\right] + 18y^2z\beta\arctan\left[\frac{y-z}{t-x}\right] - 12yz^2\beta\arctan\left[\frac{y-z}{t-x}\right] + \\
& + 2z^3\beta\arctan\left[\frac{y-z}{t-x}\right] + 24tx^2\log\left[x^2+y^2\right] - 8x^3\log\left[x^2+y^2\right] + 24ty^2\log\left[x^2+y^2\right] - 9tx^2\beta\log\left[x^2+y^2\right] + \\
& + 2x^3\beta\log\left[x^2+y^2\right] - 9ty^2\beta\log\left[x^2+y^2\right] - 24xy^2\log\left[\frac{x^2+y^2}{a^2}\right] + 6xy^2\beta\log\left[\frac{x^2+y^2}{a^2}\right] - \\
& - 8t^3\log\left[t^2-2tx+x^2+(y-z)^2\right] + 24t^2x\log\left[t^2-2tx+x^2+(y-z)^2\right] - 24tx^2\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 8x^3\log\left[t^2-2tx+x^2+(y-z)^2\right] - 24ty^2\log\left[t^2-2tx+x^2+(y-z)^2\right] + 24tz^2\log\left[t^2-2tx+x^2+(y-z)^2\right] - \\
& - 24xz^2\log\left[t^2-2tx+x^2+(y-z)^2\right] + 5t^3\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - 12t^2x\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 9tx^2\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - 2x^3\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + 9ty^2\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - \\
& - 6tyz\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] - 3tz^2\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + 6xz^2\beta\log\left[t^2-2tx+x^2+(y-z)^2\right] + \\
& + 24xy^2\log\left[\frac{t^2-2tx+x^2+(y-z)^2}{a^2}\right] - 6xy^2\beta\log\left[\frac{t^2-2tx+x^2+(y-z)^2}{a^2}\right]\Big);
\end{aligned}
$$

$$Q_{11} = \int_{-b}^{b}\int_{-a}^{a} Q_1(x,y)\,\mathrm{d}x\,\mathrm{d}y \;;$$

$$Q_{22} = \int_{-b}^{b}\int_{-a}^{a} Q_2(x,y)\,\mathrm{d}x\,\mathrm{d}y \;;$$

$Q_{33} = \int_{-b}^{b} \int_{-a}^{a} Q_3(x, y)\, \mathrm{d}x\, \mathrm{d}y$;

$Q_{44} = \int_{-b}^{b} \int_{-a}^{a} Q_4(x, y)\, \mathrm{d}x\, \mathrm{d}y$;

Using the formal software "Mathematica", this challenge was overcome, and the formulas of the coefficients were found in their simplest form:

$$
\begin{aligned}
A_{22} = \frac{1}{16\mathrm{D}1\beta} a^2 \Bigg(& \frac{t^2}{a^2} - \frac{z^2}{a^2} - \frac{1}{4a^2\pi} \beta \Bigg(-a(a-2t)\,\mathrm{ArcTan}\left[\frac{a-t}{a-z}\right] - a(a+2t)\,\mathrm{ArcTan}\left[\frac{a+t}{a-z}\right] + \left(t^2 + (a-z)^2\right) \\
& \times \mathrm{ArcTan}\left[\frac{a-z}{a-t}\right] + \left(t^2 + (a-z)^2\right) \mathrm{ArcTan}\left[\frac{a-z}{a+t}\right] - a^2 \mathrm{ArcTan}\left[\frac{a-t}{a+z}\right] + 2at\,\mathrm{ArcTan}\left[\frac{a-t}{a+z}\right] - \\
& - a^2 \mathrm{ArcTan}\left[\frac{a+t}{a+z}\right] - 2at\,\mathrm{ArcTan}\left[\frac{a+t}{a+z}\right] + a^2 \mathrm{ArcTan}\left[\frac{a+z}{a-t}\right] + t^2 \mathrm{ArcTan}\left[\frac{a+z}{a-t}\right] + 2az\,\mathrm{ArcTan}\left[\frac{a+z}{a-t}\right] + \\
& + z^2 \mathrm{ArcTan}\left[\frac{a+z}{a-t}\right] + a^2 \mathrm{ArcTan}\left[\frac{a+z}{a+t}\right] + t^2 \mathrm{ArcTan}\left[\frac{a+z}{a+t}\right] + 2az\,\mathrm{ArcTan}\left[\frac{a+z}{a+t}\right] + z^2 \mathrm{ArcTan}\left[\frac{a+z}{a+t}\right] \Bigg) \Bigg);
\end{aligned}
$$

$$
\begin{aligned}
B_{22} = \frac{1}{16\mathrm{D}1\beta} a^2 \Bigg(& \frac{2tz}{a^2} - \frac{1}{8a^2\pi} \beta \Big(-\left(2a^2 + t^2 + 2a(t-z) + z^2\right) \mathrm{Log}\left[a^2 + 2at + t^2 + (a-z)^2\right] - \\
& - \left(2a^2 - 2at + t^2 + 2az + z^2\right) \mathrm{Log}\left[2a^2 - 2at + t^2 + 2az + z^2\right] + \left(2a^2 + t^2 + z^2 - 2a(t+z)\right) \times \\
& \times \mathrm{Log}\left[2a^2 + t^2 + z^2 - 2a(t+z)\right] + \left(2a^2 + t^2 + z^2 + 2a(t+z)\right) \mathrm{Log}\left[2a^2 + t^2 + z^2 + 2a(t+z)\right] \Big) \Bigg);
\end{aligned}
$$

$$
\begin{aligned}
A_{31} = -\frac{1}{32D1(-8+\beta)} 3a^2 \Bigg(& \frac{t\left(\frac{t^2}{a^2} + \frac{z^2}{a^2}\right)}{a} - \frac{1}{24a^3\pi} \Big(4a^2 t\,(16 - 24\pi - 4\beta + 9\pi\beta) + 4a\left(12at + 2a^2\beta + 3t^2\beta\right) \arctan\left[\frac{a-t}{a-z}\right] + 2a^2(16a + 9t\beta) \arctan\left[\frac{-a+t}{a-z}\right] - \\
& - 2a\left(4a^2(-4+\beta) + 6t^2\beta + 3at(-8+3\beta)\right) \arctan\left[\frac{a+t}{a-z}\right] + 2t\left(48az + 9\left(a^2+z^2\right)\beta + t^2(8+\beta)\right) \arctan\left[\frac{a-z}{a-t}\right] - 48a^2 t \arctan\left[\frac{a-z}{a+t}\right] + \\
& + 16t^3 \arctan\left[\frac{a-z}{a+t}\right] + 96atz \arctan\left[\frac{a-z}{a+t}\right] - 48tz^2 \arctan\left[\frac{a-z}{a+t}\right] + 18a^2 t\beta \arctan\left[\frac{a-z}{a+t}\right] + 2t^3\beta \arctan\left[\frac{a-z}{a+t}\right] - 36atz\beta \arctan\left[\frac{a-z}{a+t}\right] + \\
& + 18tz^2\beta \arctan\left[\frac{a-z}{a+t}\right] + 48a^2 t \arctan\left[\frac{-a+z}{a-t}\right] + 48tz^2 \arctan\left[\frac{-a+z}{a-t}\right] + 36atz\beta \arctan\left[\frac{-a+z}{a-t}\right] - 32a^3 \arctan\left[\frac{a-t}{a+z}\right] + \\
& + 48a^2 t \arctan\left[\frac{a-t}{a+z}\right] + 8a^3\beta \arctan\left[\frac{a-t}{a+z}\right] - 18a^2 t\beta \arctan\left[\frac{a-t}{a+z}\right] + 12at^2\beta \arctan\left[\frac{a-t}{a+z}\right] + 32a^3 \arctan\left[\frac{a+t}{a+z}\right] + 48a^2 t \arctan\left[\frac{a+t}{a+z}\right] - \\
& - 8a^3\beta \arctan\left[\frac{a+t}{a+z}\right] - 18a^2 t\beta \arctan\left[\frac{a+t}{a+z}\right] - 12at^2\beta \arctan\left[\frac{a+t}{a+z}\right] - 48a^2 t \arctan\left[\frac{a+z}{a-t}\right] + 16t^3 \arctan\left[\frac{a+z}{a-t}\right] - 96atz \arctan\left[\frac{a+z}{a-t}\right] - \\
& - 48tz^2 \arctan\left[\frac{a+z}{a-t}\right] + 18a^2 t\beta \arctan\left[\frac{a+z}{a-t}\right] + 2t^3\beta \arctan\left[\frac{a+z}{a-t}\right] + 36atz\beta \arctan\left[\frac{a+z}{a-t}\right] + 18tz^2\beta \arctan\left[\frac{a+z}{a-t}\right] - \\
& - 48a^2 t \arctan\left[\frac{a+z}{a+t}\right] + 16t^3 \arctan\left[\frac{a+z}{a+t}\right] - 96atz \arctan\left[\frac{a+z}{a+t}\right] - 48tz^2 \arctan\left[\frac{a+z}{a+t}\right] + 18a^2 t\beta \arctan\left[\frac{a+z}{a+t}\right] + 2t^3\beta \arctan\left[\frac{a+z}{a+t}\right] + \\
& + 36atz\beta \arctan\left[\frac{a+z}{a+t}\right] + 18tz^2\beta \arctan\left[\frac{a+z}{a+t}\right] - 8a^3 \log\left[a^2 + 2at + t^2 + (a-z)^2\right] + 24at^2 \log\left[a^2 + 2at + t^2 + (a-z)^2\right] +
\end{aligned}
$$

$$+48a^2z\log\left[a^2+2at+t^2+(a-z)^2\right]-24t^2z\log\left[a^2+2at+t^2+(a-z)^2\right]-24az^2\log\left[a^2+2at+t^2+(a-z)^2\right]+$$

$$+8z^3\log\left[a^2+2at+t^2+(a-z)^2\right]+2a^3\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-6at^2\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-$$

$$-18a^2z\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-6atz\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+3t^2z\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+$$

$$+12az^2\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-5z^3\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-24a^3\log\left[\frac{a^2+2at+t^2+(a-z)^2}{a^2}\right]+$$

$$+6a^3\beta\log\left[\frac{a^2+2at+t^2+(a-z)^2}{a^2}\right]+8a^3\log\left[2a^2-2at+t^2+2az+z^2\right]-24at^2\log\left[2a^2-2at+t^2+2az+z^2\right]+$$

$$+48a^2z\log\left[2a^2-2at+t^2+2az+z^2\right]-24t^2z\log\left[2a^2-2at+t^2+2az+z^2\right]+24az^2\log\left[2a^2-2at+t^2+2az+z^2\right]+$$

$$+8z^3\log\left[2a^2-2at+t^2+2az+z^2\right]-2a^3\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+6at^2\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-$$

$$-18a^2z\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+6atz\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+3t^2z\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-$$

$$-12az^2\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-5z^3\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+24a^3\log\left[\frac{2a^2-2at+t^2+2az+z^2}{a^2}\right]-$$

$$-6a^3\beta\log\left[\frac{2a^2-2at+t^2+2az+z^2}{a^2}\right]+8a^3\log\left[2a^2+t^2+z^2-2a(t+z)\right]-$$

$$-24at^2\log\left[2a^2+t^2+z^2-2a(t+z)\right]-48a^2z\log\left[2a^2+t^2+z^2-2a(t+z)\right]+24t^2z\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$

$$+24az^2\log\left[2a^2+t^2+z^2-2a(t+z)\right]-8z^3\log\left[2a^2+t^2+z^2-2a(t+z)\right]-2a^3\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$

$$+6at^2\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+18a^2z\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-6atz\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-$$

$$-3t^2z\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-12az^2\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+5z^3\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$

$$+24a^3\log\left[\frac{2a^2+t^2+z^2-2a(t+z)}{a^2}\right]-6a^3\beta\log\left[\frac{2a^2+t^2+z^2-2a(t+z)}{a^2}\right]-8a^3\log\left[2a^2+t^2+z^2+2a(t+z)\right]+$$

$$+24at^2\log\left[2a^2+t^2+z^2+2a(t+z)\right]-48a^2z\log\left[2a^2+t^2+z^2+2a(t+z)\right]+24t^2z\log\left[2a^2+t^2+z^2+2a(t+z)\right]-$$

$$-24az^2\log\left[2a^2+t^2+z^2+2a(t+z)\right]-8z^3\log\left[2a^2+t^2+z^2+2a(t+z)\right]+2a^3\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]-$$

$$-6at^2\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+18a^2z\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+6atz\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]-$$

$$-3t^2z\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+12az^2\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+5z^3\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+$$

$$+6a^3(-4+\beta)\log\left[\frac{2a^2+t^2+z^2+2a(t+z)}{a^2}\right]\Bigg]\Bigg)\Bigg);$$

$$B_{31}=-\frac{1}{32D1(-8+\beta)}3a^2\left(\frac{z\left(\frac{t^2}{a^2}+\frac{z^2}{a^2}\right)}{a}-\frac{1}{24a^3\pi}\Big(4a^2z\,(16+24\pi-4\beta-9\pi\beta)+6a(a-2t)z\,(-8+3\beta)\arctan\left[\frac{a-t}{a-z}\right]+\right.$$

$$+6a(a+2t)z\,(-8+3\beta)\arctan\left[\frac{a+t}{a-z}\right]+2\left(3a^2z\,(8-3\beta)+4a^3(-4+\beta)+6az^2\beta-z\left(z^2(8+\beta)+3t^2(-8+3\beta)\right)\right)\arctan\left[\frac{a-z}{a-t}\right]+$$

$$+2\left(3a^2z\,(8-3\beta)+4a^3(-4+\beta)+6az^2\beta-z\left(z^2(8+\beta)+3t^2(-8+3\beta)\right)\right)\arctan\left[\frac{a-z}{a+t}\right]-48a^2z\arctan\left[\frac{a-t}{a+z}\right]+96atz\arctan\left[\frac{a-t}{a+z}\right]+$$

$$+18a^2z\beta\arctan\left[\frac{a-t}{a+z}\right]-36atz\beta\arctan\left[\frac{a-t}{a+z}\right]-48a^2z\arctan\left[\frac{a+t}{a+z}\right]-96atz\arctan\left[\frac{a+t}{a+z}\right]+18a^2z\beta\arctan\left[\frac{a+t}{a+z}\right]+$$

$$+36atz\beta\arctan\left[\frac{a+t}{a+z}\right]+32a^3\arctan\left[\frac{a+z}{a-t}\right]+48a^2z\arctan\left[\frac{a+z}{a-t}\right]+48t^2z\arctan\left[\frac{a+z}{a-t}\right]-16z^3\arctan\left[\frac{a+z}{a-t}\right]-8a^3\beta\arctan\left[\frac{a+z}{a-t}\right]-$$

$$-18a^2z\beta\arctan\left[\frac{a+z}{a-t}\right]-18t^2z\beta\arctan\left[\frac{a+z}{a-t}\right]-12az^2\beta\arctan\left[\frac{a+z}{a-t}\right]-2z^3\beta\arctan\left[\frac{a+z}{a-t}\right]+32a^3\arctan\left[\frac{a+z}{a+t}\right]+48a^2z\arctan\left[\frac{a+z}{a+t}\right]+$$

$$+48t^2z\arctan\left[\frac{a+z}{a+t}\right]-16z^3\arctan\left[\frac{a+z}{a+t}\right]-8a^3\beta\arctan\left[\frac{a+z}{a+t}\right]-18a^2z\beta\arctan\left[\frac{a+z}{a+t}\right]-18t^2z\beta\arctan\left[\frac{a+z}{a+t}\right]-12az^2\beta\arctan\left[\frac{a+z}{a+t}\right]-$$

$$-2z^3\beta\arctan\left[\frac{a+z}{a+t}\right]+8a^3\log\left[a^2+2at+t^2+(a-z)^2\right]+48a^2t\log\left[a^2+2at+t^2+(a-z)^2\right]+24at^2\log\left[a^2+2at+t^2+(a-z)^2\right]+$$

$$+8t^3\log\left[a^2+2at+t^2+(a-z)^2\right]-24az^2\log\left[a^2+2at+t^2+(a-z)^2\right]-24tz^2\log\left[a^2+2at+t^2+(a-z)^2\right]-2a^3\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-$$

$$-18a^2t\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-12at^2\beta\log\left[a^2+2at+t^2+(a-z)^2\right]-5t^3\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+$$
$$+6atz\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+6az^2\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+3tz^2\beta\log\left[a^2+2at+t^2+(a-z)^2\right]+$$
$$+24a^3\log\left[\frac{a^2+2at+t^2+(a-z)^2}{a^2}\right]-6a^3\beta\log\left[\frac{a^2+2at+t^2+(a-z)^2}{a^2}\right]-8a^3\log\left[2a^2-2at+t^2+2az+z^2\right]+$$
$$+48a^2t\log\left[2a^2-2at+t^2+2az+z^2\right]-24at^2\log\left[2a^2-2at+t^2+2az+z^2\right]+8t^3\log\left[2a^2-2at+t^2+2az+z^2\right]+$$
$$+24az^2\log\left[2a^2-2at+t^2+2az+z^2\right]-24tz^2\log\left[2a^2-2at+t^2+2az+z^2\right]+2a^3\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-$$
$$-18a^2t\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+12at^2\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-5t^3\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-$$
$$-6atz\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-6az^2\beta\log\left[2a^2-2at+t^2+2az+z^2\right]+3tz^2\beta\log\left[2a^2-2at+t^2+2az+z^2\right]-$$
$$-24a^3\log\left[\frac{2a^2-2at+t^2+2az+z^2}{a^2}\right]+6a^3\beta\log\left[\frac{2a^2-2at+t^2+2az+z^2}{a^2}\right]+8a^3\log\left[2a^2+t^2+z^2-2a(t+z)\right]-$$
$$-48a^2t\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$

$$+24at^2\log\left[2a^2+t^2+z^2-2a(t+z)\right]-8t^3\log\left[2a^2+t^2+z^2-2a(t+z)\right]-24az^2\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$
$$+24tz^2\log\left[2a^2+t^2+z^2-2a(t+z)\right]-2a^3\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+18a^2t\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-$$
$$-12at^2\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+5t^3\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-6atz\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+$$
$$+6az^2\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]-3tz^2\beta\log\left[2a^2+t^2+z^2-2a(t+z)\right]+24a^3\log\left[\frac{2a^2+t^2+z^2-2a(t+z)}{a^2}\right]-$$
$$-6a^3\beta\log\left[\frac{2a^2+t^2+z^2-2a(t+z)}{a^2}\right]-8a^3\log\left[2a^2+t^2+z^2+2a(t+z)\right]-48a^2t\log\left[2a^2+t^2+z^2+2a(t+z)\right]-$$
$$-24at^2\log\left[2a^2+t^2+z^2+2a(t+z)\right]-8t^3\log\left[2a^2+t^2+z^2+2a(t+z)\right]+24az^2\log\left[2a^2+t^2+z^2+2a(t+z)\right]+$$
$$+24tz^2\log\left[2a^2+t^2+z^2+2a(t+z)\right]+2a^3\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+18a^2t\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+$$
$$+12at^2\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+5t^3\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+6atz\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]-$$
$$-6az^2\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]-3tz^2\beta\log\left[2a^2+t^2+z^2+2a(t+z)\right]+6a^3(-4+\beta)\log\left[\frac{2a^2+t^2+z^2+2a(t+z)}{a^2}\right]\Bigg]\Bigg)\Bigg);$$

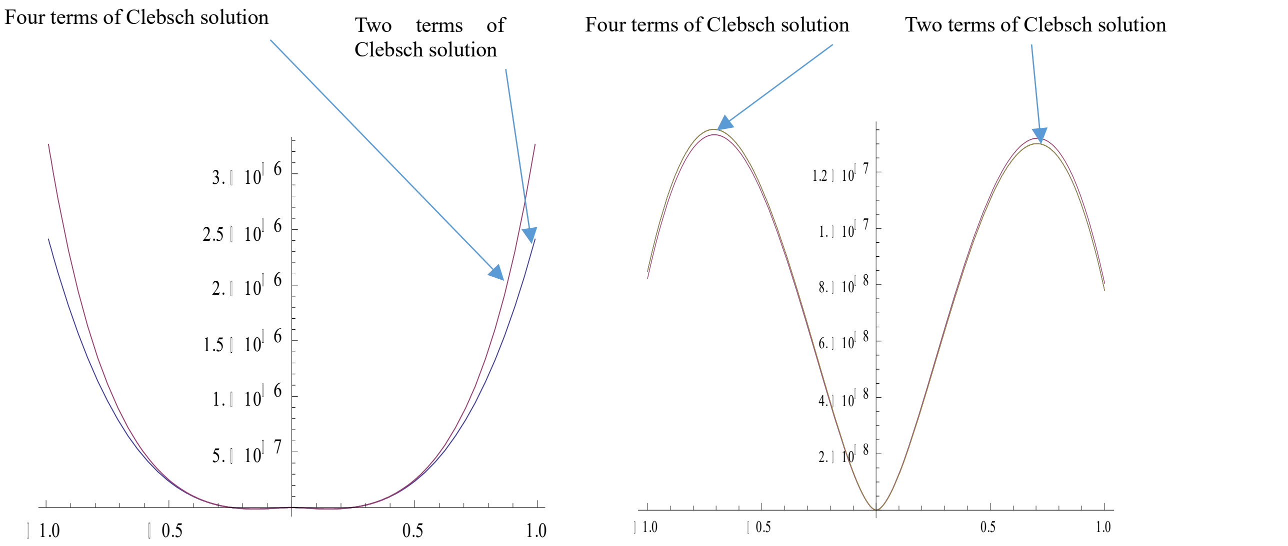
Proof of The Number of Terms Taken

To justify the number of terms of the Clebsch solution taken for this case, the following verification is provided. First, only two terms of Clebsch's solution are considered with respect to the special solution W_0. The given displacement values of the diagonal points of the plate are considered with respect to the following geometric and mechanical characteristics ($a = b = 1m$; $\nu = 1/3$; $E = 2 \times 10^{11} N/m^2$; $h = 0.02m$).

(i)- when the plate is loaded uniformly over its surface with an external force $q = 1KN/m^2$, the displacements are illustrated in figure 1, (a).

(ii)- when the external force $P = 1KN$ is applied at the plate's center, the displacements are illustrated in figure 1, (b).

The same operation is now repeated with four numbers of terms of the Clebsch solution. The results clearly demonstrate that the displacement values of the diagonal of the plate are practically the same for both cases. This justifies the number of terms of the Clebsch solution taken for the determination of the deflections of the plate for the studied case

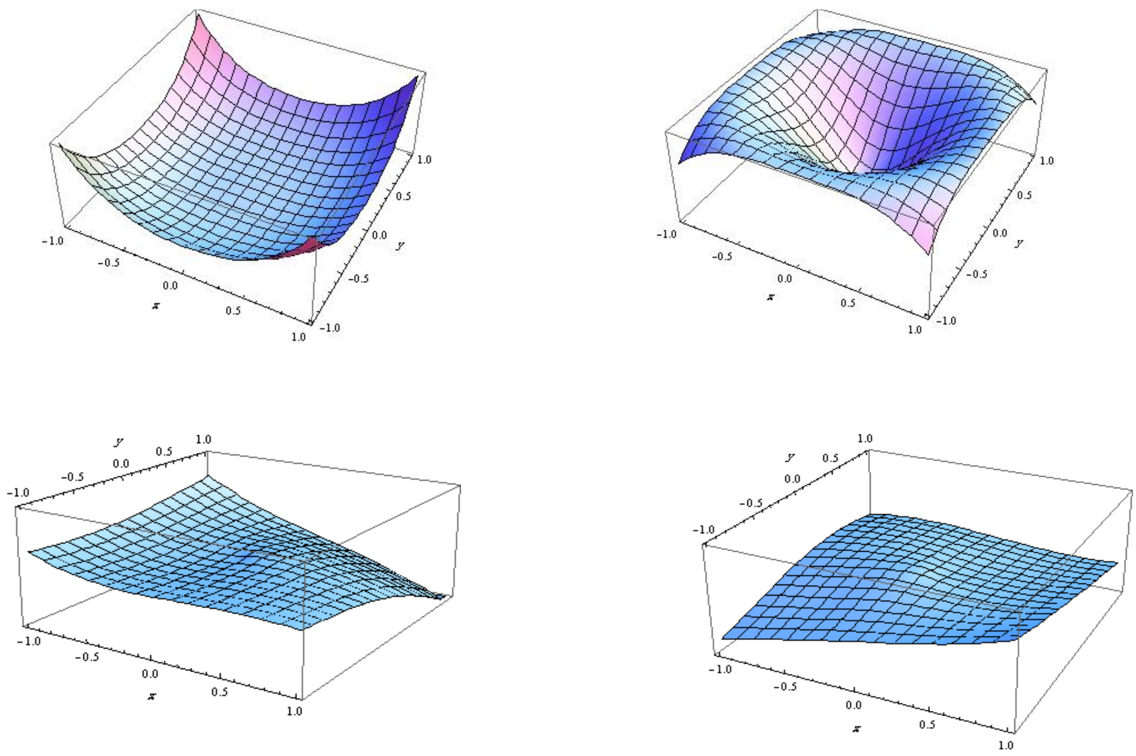


(a) (b)

Figure 1. Comparison of the obtained results taking into account different numbers of terms of the Clebsch solution

a. Plate loaded by an uniformly load over the entire surface of the plate
b. Plate loaded by a concentrate load applied at its center

The general shapes of the deflected plate in three-dimensions (3D) due to the different external load cases are illustrated in Figure 2.



(c) (d)

Figure 2. Forms of the plate deflections due to different cases of external loads

a. Plate loaded by an uniformly load over the entire surface of the plate
b. Plate loaded by a concentrate load applied at the center of the plate
c. Plate loaded by a concentrate load applied at a point of coordinates x=0.9 and y=0.9
d. Plate loaded by a concentrate load applied at a point of coordinates x=0.5 and y=0.5

Results and Discussion

From the 3D graphs in figure 2, the plate deflections are shown to be in good agreement with the natural deflection case. For example in the case of the concentrated load at the center of the plate, the coherent distribution of the deflections of the plate where the maximum displacement at the center of the plate are clearly visible. The same coinciding outputs are observed for the other loadings and distributions, especially for the external uniformly distributed load over the plate's entire surface. The solution given by our approach permits to determine the value of the function of the displacements of the studied plate with any geometrical dimensions at any point and due to any type of the external load. Moreover, it's given in the algorithm format allowing its application easy.

Concluding Remarks

This paper presents a semi-analytical resolution of the fourth-order differential equation of the deflections of a thin rectangular sqare plate resting on the surface of an elastic foundation. To accomplish this task, Clebsch's solution applied to this problem. This solution is associated with coefficients to be determined by the application of the energy method of Ritz and with the application of the boundary conditions of the studied plate. During the accomplishment of this work, a major mathematical challenge overcome through use of "Mathematica", specifically identification of the expressions of the searched coefficients. The resulting algorithm can be applied not only in the dynamic study of a contact problem of a rectangualr plate resting on the surface of an elastic medium of inertial properties (Lamb model), but also for any rectangular plate across multiple fields such as construction and manufacturing.

Nomenclature

$W(x,y)$	Function of displacements of the plate
Δ	Laplace operator
$q(x,y)$	Function of the external load subjected on the plate
D	Cylindrical rigidity of the plate.
E	Elasticity modulus of the plate's materials
ν	Poisson's ratio of the plate's materials
U	The deformation energy of the plate
Π	The work of deformation of the plate
$\in$	The total deformation energy of the plate
P	External load applied at any point of the plate

a	length of the plate
b	width of the platet and
z	coordinates of the point where the external load is applied

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